OCCURRENCE OF SPACE-PERIODIC MOTIONS IN HYDRODYNAMICS

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A nonlinear equation is derived for the amplitude of a one-dimensional unsteady flow resulting from a disturbance in the stability of the primary flow. The latter is also assumed to be one-dimensional and independent of t and the spatial coordinate. Supercriticality is assumed to be small and the wave number spectrum to be discrete, though arbitrarily dense. The equation is simplified under the assumption that the secondary flow can be represented using the process of superposition of the wave packets with multiple wave numbers (accumulating perturbations form a narrow wave packet). The equation has a large number of stable steady-state solutions differing from each other by their wavelengths, and a unsteady problem is solved in order to obtain the value of the wavelength. The latter solution is obtained in the form of a series in the terms of the initial amplitude, and converges when t is finite. When t is large, the series is summed and validity of the solution is thus extended to the values of t at which the series diverges. We find that a periodic motion is established in the system and, that its wavelength characterizes the perturbation with the largest increment. A double periodic turbulent motion is established for discrete values of the parameter (for which the largest increment possesses two perturbations).

1. It is well known that, when the steady-state becomes unstable, a periodic motion of amplitude Q whose modulus satisfies [1 to 6]

$$dq / dt = 2q \ (\gamma + aq + bq^2 + \ldots), \qquad q = | Q |^2 \tag{1.1}$$

may be set up in the system. In this equation the increment γ of the accumulating (in the linear theory) perturbation and the magnitudes a and b (connected with the nonlinear terms), are functions of the parameters λ; when λ is critical, i.e. λ_*, $\gamma = 0$.

Eq. (1.1) was derived under the assumption that the spectrum of eigenvalues of the linearized boundary value problem is discrete [1]. This is the case when e.g. the fluid moves in a limited volume. In this paper we consider bounded systems, in which the longitudinal dimension l is large compared with the transverse dimension (e.g. in the problem on the flow of fluid between two rotating cylinders [7] the length l of the cylinders is assumed large compared to the gap between them; again, in the investigation of the positive gas discharge column [8], the length of the column was assumed large compared with the radius of the discharge tube). When the steady-state and stability of such systems are investigated, the end effects are neglected and the length is assumed to be infinite ($l = \infty$). This implies that the steady-state parameters are independent of the longitudinal x-coordinate, while the eigenfunctions of the problem of stability of the steady-state are proportional to exp (ikx) where the wave number k may assume any real value. For any k there exists an infinite set (branch) of eigenvalues $p = \gamma + i\Omega$ and each eigenvalue considered as a function of k, defines a continuous set (branch) of eigenvalues.

Fig. 1 illustrates the typical case of the onset of instability following the change in the parameter value; the broken line shows the decrement of one of the stable branches (which characterize only the decaying perturbations). The increment of the accumulating perturba-

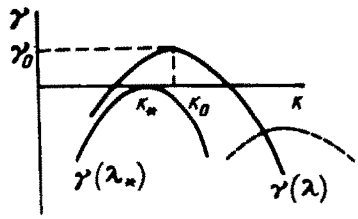

Fig. 1

tions will be maximum when

$$k = k_0(\lambda) \tag{1.2}$$

$$\gamma(k) = \gamma_0 + {}^1/_2\gamma_0''(k - k_0)^2 \qquad (\gamma_0'' < 0)$$

where, as usual, $1/k_0$ is of the order of the transverse dimension of the system.

Supercriticality $\Lambda = \lambda - \lambda_*$ can be defined by the half-width of the Δ-interval in which $\gamma(k) > 0$

$$\Delta = \sqrt{-2\gamma_0/\gamma_0''} \sim \sqrt{\Lambda} \tag{1.3}$$

In the following we shall always assume the supercriticality to be sufficiently small to ensure that $\Delta \ll k_0$.

The state of the systems under consideration is defined in terms of the parameters of the infinite problem ("longitudinal" boundary conditions influence this state only near the ends $x = \pm\frac{1}{2}l$). Therefore in the following the boundedness of the system is only reflected in the fact that the deviations of the parameters from their equilibrium values can be given in terms of a Fourier series in x. Consequently we find that the wave numbers assume the following discrete values

$$k = 2\delta n, \quad \delta = \pi/l \quad (n = 0, \pm 1, \ldots) \tag{1.4}$$

in all relations $f(k)$ of the infinite problem. Here k_0 defines approximately the wave number $\varkappa$ of the discrete spectrum, associated with the largest increment $|\varkappa - k_0| \leqslant \delta$.

The last expression implies that the approximation $l = \infty$ is applicable when $\delta \ll k_0$. In the following we shall assume that this condition holds, although the relation between δ and the half-width Δ in (1.3) may be arbitrary.

If $\Delta < \delta$, then only one perturbation accumulates in the system. Its wave number is $\varkappa \approx k_*$ and its amplitude is given by (1.1). When $a < 0$, then a steady, space-periodic motion whose wave number is $\varkappa \approx k_*$, is set up in the system.

If the supercriticality is large enough to ensure that

$$\delta \lll \Delta \lll k_0 \tag{1.5}$$

then a large number of perturbations (infinite if $l = \infty$) accumulates in the system. In this case it is not at all obvious that the small amplitude motion which vanishes as $\Lambda \to 0$, should be space-periodic. If we accept this as an empirical datum, then the theoretical determination of the amplitude and the wave number of the steady-state periodic motion, will yield only one Eq. (1.1) in which the coefficients will be known functions of parameters and of the wave number [5 and 6]. Thus the wave number of the steady-state solutions will remain an undefined parameter of the theory [9, 5 and 6].

To remove this indeterminacy, an equation (2.15) was obtained and solved in this paper. It describes the interaction of a large number of accumulating perturbations.

2. In this Section we shall derive a method of obtaining the amplitude equation for the accumulating perturbations from the initial hydrodynamic equations of the type

$$dX/dt = F(X, d(\ldots)/dx, d(\ldots)/dr, r, \lambda) \tag{2.1}$$

where the vector X denotes the set of hydrodynamic variables (density, temperature, magnetic field etc.), x is the longitudinal coordinate and r denotes the set of "transverse" coordinates.

We can assume without the loss of generality that the equilibrium state (independent of t and x) is given by $X = 0$ and that the boundary conditions are linear in X, homogeneous and do not contain any derivatives of X with respect to t [5].

Initial deviation from the equilibrium state is assumed small

$$X(x, t = 0) = \varepsilon X_0(x)$$

Here the amplitude $\varepsilon \to 0$ and the function X_0 which describes the form of the deviation is normalized in some manner. In the following we investigate the cases when X remains small at any instant of time and when (2.1) can be expanded into

$$\frac{\partial X}{\partial t} = LX + \sum_{n=2}^{\infty} (L_1{}^n X) \dots (L_n{}^n X) \tag{2.2}$$

where the matrices L are, unlike F, independent of X and the boundary condition has the form $L_0 X = 0$. Solution X is sought in the form

$$X = \sum_k Y(k)\, e^{ikx}, \qquad Y(-k) = \overline{Y}(k) \tag{2.3}$$

where k assumes the values given by (1.4) and the bar denotes a complex conjugate. Inserting (2.3) into (2.2) we obtain the following equation for $Y(k)$

$$\frac{\partial Y}{\partial t} - L(k)\, Y = \sum_{n=2}^{\infty} \sum_k [L_1{}^n (k_1)\, Y(k_1)] \dots [L_n{}^n (k_n)\, Y(k_n)] \tag{2.4}$$

$$Y(t = 0) = \varepsilon Y_0, \qquad L_0(k)\, Y = 0 \qquad (k_1 + \dots + k_n = k)$$

Here the matrices $L(k)$ are obtained from the corresponding matrices appearing in (2.2), by means of the substitution $d(\dots)/dx \to ik$. Solution Y is sought in the form (*)

$$Y(k, r, \lambda) = QZ(k, r, \lambda) + \sum_{n=2}^{\infty} \sum_k Z_n(k_1, \dots, k_n; r, \lambda)\, Q(k_1) \dots Q(k_n) \tag{2.5}$$

where the amplitude $Q(k, t)$ satisfies Eq.

$$\frac{dQ}{dt} = pQ + \sum_{n=2}^{\infty} \sum_k H_n(k_1, \dots, k_n; \lambda)\, Q(k_1) \dots Q(k_2) \tag{2.6}$$

and the magnitudes p, Q, Z and H become their complex conjugates on change of the sign of the wave numbers.

Eq. (2.4) after the insertion of (2.5) and (2.6) becomes

$$QD + \sum_{n=2}^{\infty} \sum_k Q(k_1) \dots Q(k_n)\, D_n(k_1, \dots; k_n) = 0$$

Values of Z_n and H_n are found, consecutively, from Eqs. $D_n = 0$ together with the boundary condition $L_0 Z_n = 0$. The linear problem

$$D \equiv pZ - LZ = 0, \qquad L_0 Z = 0 \tag{2.7}$$

defines the equilibrium stability. We assume that the onset of instability follows the course shown in Fig. 1. Parameters of the system are assumed to be such, that (1.5) holds.

In (2.5) and (2.6) the eigenvalue $p = \gamma + i\Omega$ and the eigenfunction Z of the problem (2.7), both defining the accumulating perturbations, must be used.

In the nonhomogeneous problem

$$D_n \equiv Z_n P_n - LZ_n + H_n Z + \Psi_n = 0, \quad L_0 Z_n = 0, \quad P_n = p(k_1) + \dots + p(k_n) \tag{2.8}$$

the vector Ψ_n is expressed in the terms of previously found H and Z.

Solution Z_n can be obtained [10] with help of the Green's matric $G(r, \rho, P_n)$ of the homogeneous problem (2.8)

$$Z_n = \int_S G(r, \rho, P_n)\, [H_n Z(\rho) + \Psi_n(\rho)]\, d\rho$$

where the integration with respect to transverse coordinates ρ is performed over the transverse section S of the system. The Green's matrix can be represented by [10]

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*) Here and in the next sum, wave numbers appear which satisfy the condition $k_1 + \dots + k_n = k$.

$$G = -\frac{Z\overline{U}}{(P_n - p)\langle Z\cdot U\rangle} + G_- \tag{2.9}$$

where G_- is regular when $P_n = p$, U is the eigenfunction of the problem conjugate to (2.7) corresponding to the eigenvalue $\bar{p}$ and the scalar product is

$$\langle Z\cdot U\rangle = \int_S (Z\cdot\overline{U})\, d\rho$$

The difference $P_n - p$ may become very small, e.g. if $\gamma(k) = 0$, then $P_3 = p$ when $k_1 = k_2 = -k_3 = k$.

In accordance with (2.8) and (2.9), the vector Z_n will be finite for any k, if

$$H_n \langle Z\cdot U\rangle + \langle \Psi_n \cdot U\rangle = 0, \qquad Z_n = \int_S G_- (H_n Z + \Psi_n)\, d\rho$$

Here the first equation defines H_n, while the other defines Z_n.

Let γ_1 be a minimal decrement of perturbations associated with stable branches. We may expect that when $\varepsilon \to 0$, then (2.5) and (2.6) describe the behavior of the system beginning at the instant $t \sim 1/(\gamma_0 + \gamma_1)$, when, in accordance with the linear theory, only those perturbations are essential which are associated with the unstable branch. Consequently we may take

$$Q\ (t = 0) = \varepsilon A\ (k) \tag{2.10}$$

as the initial condition for (2.6). Here A is the component of the initial vector Y_0 corresponding to the accumulating perturbations

$$A\langle Z\cdot U\rangle = \langle Y_0 \cdot U\rangle$$

Generally speaking, Eqs. (2.5) and (2.6) are more accurate than the approximation in which the perturbations with large decrement given at $t = 0$ are neglected. For example, in the case illustrated in Fig. 1, the initial perturbation amplitudes of the unstable branch for large k need not be taken into account, since their decrements exceed the perturbation decrements of one of the stable branches. We can remove this excess of accuracy by reducing (2.6) to an equation for accumulating perturbations with the wave number $|k| \approx k_0$; these perturbations will form a wave packet of the bandwidth equal to 2Δ.

By virtue of the condition $\Delta \ll k_0$, nonlinear effect in the spectrum $Q(k)$ causes the separation of the additional wave packets whose effective wave number is nk_0 (where n is an integer). The amplitude of these packets can be expressed in the functional form in terms of the amplitudes of the fundamental packets

$$Q\ (k \approx nk_0) = \sum_{m=0}^{\infty} \sum h_{n,\, n+2m} Q_1 \ldots Q_{n+2m} \tag{2.11}$$

while the equation for $Q(k \approx k_0)$ has the form

$$\frac{dQ}{dt} = pQ + \sum_{m=0}^{\infty} \sum h_{1,\, 1+2m} Q_1 \ldots Q_{1+2m} \tag{2.12}$$

Here and in the following we use the following abbreviation

$$\sum f Q_1 \ldots Q_n = \sum_k f\,(k_1, \ldots, k_n)\, Q\,(k_1) \ldots Q\,(k_n), \qquad |k_i| \approx k_0$$

and h become complex conjugates when k change their sign.

To find $h_{n,\, n+2m}$ we must put $k \approx nk_0$ in (2.6) and assume in the summation performed over k, that $k_i \approx m_i k_0$ where m is an integer. Inserting (2.11) and (2.12) into (2.6), we obtain

$$\sum_{m=0}^{\infty} \sum \Phi_{n,\, n+2m} Q_1 \ldots Q_{n+2m} = 0$$

while the linear algebraic equation

$$\Phi_{n,n+2m} = 0$$

yields $h_{n,n+2m}$.

Determination of $h_{n,n+2m}$ must however be preceded by the determination of all h appearing in the matrix to the left of the diagonal drawn through $h_{n,n+2m+2}$ (part of the matrix is shown below)

$$\begin{matrix} \cdots & h_{02} & \cdots & h_{04} & \cdots \\ p & \cdots & h_{13} & \cdots & h_{15} \\ \cdots & h_{22} & \cdots & h_{24} & \cdots \\ \cdots & \cdots & h_{33} & \cdots & h_{35} \\ \cdots & \cdots & \cdots & h_{44} & \cdots \end{matrix}$$

The wave packets are obtained from $Y(k)$ in the similar manner. Relations (2.5) and (2.11) yield

$$Y(k \approx nk_0) = \sum_{m=0}^{\infty} \sum V_{n,\,n+2m} Q_1 \cdots Q_{n+2m} \tag{2.13}$$

where the magnitudes V are expressed in terms of Z and h.

We can assume that the functions $f(k_1, \ldots, k_m)$ appearing in the sums of the form $\Sigma f Q_1, \ldots Q_m$ are symmetric, since the sum should not change under the permutation of $k_1, \ldots, k_m$. We use this property together with the relation $Q(-k) = \overline{Q}(k)$ to reduce (2.12) and (2.13) to

$$Y(k \approx nk_0) = \sum_{m=0}^{\infty} \sum Y_{n,\,n+2m} Q_1 \cdots Q_{n+m} \overline{Q}_{n+m+1} \cdots \overline{Q}_{n+2m} \tag{2.14}$$

$$\frac{dQ}{dt} = pQ + \sum_{m=1}^{\infty} \sum \Gamma_m Q_1 \cdots Q_{m+1} \overline{Q}_{m+2} \cdots \overline{Q}_{1+2m} \tag{2.15}$$

$$Y_{n,\,n+2m} = \frac{(n+2m)!}{m!\,(n+m)!} V(k_1, \ldots, k_{n+m}, -k_{n+m+1}, \ldots, -k_{n+2m})$$

$$\Gamma_m = (2m+1)\, h_{1,\,1+2m} (k_1, \ldots, k_{1+m}, -k_{2+m}, \ldots, -k_{1+2m})$$

In (2.14) the summation over k is performed for the values satisfying

$$k_1 + \ldots + k_{n+m} - k_{n+m+1} - \ldots - k_{n+2m} = k, \qquad k_i \approx k_0 \tag{2.16}$$

which, at $n = 1$, yield the condition for the sums over k in (2.15).

3. We shall first consider the stability of a steady-state periodic solution whose wave number is k. This solution is defined by (2.14) and (2.15) in which only the amplitude $Q(k)$ is different from zero; similar relations were obtained in [2, 3 and 5].

Eq. (2.15) for the steady-state amplitude $Q(k)$ has the form

$$\frac{dQ}{dt} = Q\,[p(k) + \sum_{m=1}^{\infty} \omega_m(k, k)\, q^m] \tag{3.1}$$

$$\gamma(k) + \sum_{m=1}^{\infty} \gamma_m(k, k)\, q^m = 0, \qquad q = Q\overline{Q} \tag{3.2}$$

Here and in the following

$$\omega_m(k', k) = \Gamma_m(k', \underbrace{k, \ldots, k}_{2m}) = \gamma_m + i\Omega_m$$

Let us now assume that the amplitude distribution differs from the steady-state distribution (in which only $Q(k) \neq 0$) by the infinitesimal perturbations $Q^\circ(k')$; then (2.15) can be linearized with respect to these perturbations to obtain, taking into account (2.16) and the symmetry of Γ,

$$\frac{dQ^\circ(k')}{dt} = Q^\circ\left[p(k') + \sum_{m=1}^{\infty}(m+1)\,\omega_m(k', k)\,q^m\right] \tag{3.3}$$

From (3.3) it follows that the periodic solution with the wave number k will be stable, if, for any k',

$$U(k', k) = \gamma(k') + \sum_{m=1}^{\infty}\gamma_m(k', k)(m+1)\,q^m < 0 \tag{3.4}$$

where $q(k)$ is given by (3.2).

By (2.16), we consider only whose wave numbers which differ from k_0 by the amounts $\sim \Delta$. Since the difference $k' - k \sim \Delta$ is small, we can write (3.4) with (3.2) taken into account, as

$$U \approx q\sum_{m=1}^{\infty} mq^m\gamma_m + (k'-k)\frac{\partial U}{\partial k'} + \frac{1}{2}(k'-k)^2\frac{\partial^2 U}{\partial (k')^2} < 0 \tag{3.5}$$

Here and below, functions of k' are assumed to be taken at $k' = k$.

Let $\gamma_1(k_0, k_0) < 0$. Then (3.2), (3.5) and (1.3) yield

$$q(k) \approx -\gamma/\gamma_1, \quad |k - k_0| \leqslant \Delta \tag{3.6}$$

$$U \approx q\gamma_1 + (k'-k)\,\gamma' + {}^1/_2\,(k'-k)^2\,\gamma'' =$$
$$= -\gamma_0 - \gamma_0''\,(k-k_0)^2 + {}^1/_2\gamma_0''\,(k'-k_0)^2 \tag{3.7}$$

By (3.7), the condition of stability $U < 0$ holds for periodic solutions whose wave number k satisfies the condition $|k - k_0| < \Delta/\sqrt{2}$. Solutions for which $\Delta \geqslant |k - k_0| > \Delta/\sqrt{2}$, are unstable with respect to the perturbations whose wave numbers k' are given by

$$(k'-k)^2 < 2\,(k-k_0)^2 - \Delta^2$$

Although investigation of the periodic solution stability strengthens restriction (3.6), it does not yield the exact value of the wavelength. The problem whether a periodic solution is established when the number of initial perturbations becomes large, remains unsettled. This can only be found in the course of solving the initial problem (2.15) and (2.10).

4. We shall investigate, for simplicity, the behavior of the principal term $X \approx 2\mathrm{Re}X_{11}$ as $t \to \infty$, where

$$X_{11} = \sum_{k \approx k_0} QZe^{ikx} \tag{4.1}$$

and Q satisfies Eq. (2.15) in which only the lowest nonlinear term is retained (for this reason the index of Γ_1 shall henceforth be omitted).

Solution of (2.15), (2.10) is sought in the form of a series in terms of the initial amplitude ε

$$Q = \sum_{n=0}^{\infty}\varepsilon^{2n+1}Q^{(n)} \tag{4.2}$$

We easily obtain

$$Q^{(0)} = Ae^{pt}, \qquad Q^{(n)} = e^{pt}\int_0^t dt e^{-pt}\sum\sum_m \Gamma Q_1^{(m_1)}Q_2^{(m_2)}\bar{Q}_3^{(m_3)} \tag{4.3}$$

where the summation is performed over k_j and over the nonnegative integers m satisfying the conditions

$$k_1 + k_2 - k_3 = k, \qquad m_1 + m_2 + m_3 = n - 1 \tag{4.4}$$

From (4.3) we can obtain $Q^{(n)}$ for large t. Let us consider

$$Q^{(1)} = e^{pt} \sum \Gamma A_1 A_2 \bar{A}_3 \frac{e^{(P-p)t} - 1}{P - p}, \qquad A_l = A(k_l)$$
$$P = p(k_1) + p(k_2) + \bar{p}(k_3) \tag{4.5}$$

Inserting (4.2) and (4.5) into (4.1) we find, that for large t, those terms of (4.5) will contribute most to X_{11}, for which

$$t \operatorname{Re}(P - p) \gg 1$$

In these terms the unity is small compared with the exponential part and can be neglected, leaving only strong exponential dependence on the wave numbers. Below we shall see that in the factors independent of t, all those k_l should be taken into account which are equal to the wave number $\varkappa$ at the maximum value of γ.

Thus for $\gamma_0 t \gg 1$ we have

$$Q^{(1)} = \Gamma A \frac{|A|^2}{2\gamma} \sum \exp tP$$

Similarly we can find that when $\gamma_0 t \gg 1$, then (4.3) is satisfied by

$$Q^{(n)} = f_n A \left(\frac{|A|^2}{2\gamma}\right)^n \sum \exp tP_n \tag{4.6}$$

$$P_n = p(k_1) + \ldots + p(k_{n+1}) + \bar{p}(k_{n+2}) + \ldots + \bar{p}(k_{2n+1})$$

where the summation is performed over k_l satisfying

$$k_1 + \ldots + k_{n+1} - k_{n+2} - \ldots - k_{2n+1} = k \tag{4.7}$$

In the terms preceding the sum, all wave numbers are equal to $\varkappa$, while the coefficients f are given by the following recurrent relation:

$$f_n = \frac{\Gamma}{n} \sum_m f_{m_1} f_{m_2} \bar{f}_{m_3}, \qquad f_0 = 1 \tag{4.8}$$

in which the summation is performed over those m, which satisfy the second condition of (4.4).

Relations (4.6), (4.2) and (4.1) yield

$$X_{11} = \sum_{n=0}^{\infty} A\varepsilon f_n \frac{|\varepsilon A|^{2n}}{(2\gamma)^n} \sum e^{ikx} Z(k) e^{P_n t} \tag{4.9}$$

where the sum is taken over the arbitrary numbers $k_1, \ldots, k_{2n+1}$ and k is defined by (4.7). Asymptotic behavior of the second sum of (4.9) depends essentially on t. If $|\gamma_0''| \delta^2 t \ll 1$ (which does not contradict the condition $\gamma_0 t \gg 1$ since $\delta \ll \Delta$), then the sums with similar k differ little from each other and can be replaced by integrals over the wave numbers (see Appendix); for such t the system behaves like an infinite one.

In the opposite limiting case

$$t(\gamma - \gamma_1) \gg 1, \qquad \gamma = \gamma(\varkappa), \qquad \gamma_1 = \max[\gamma(\varkappa - 2\delta), \gamma(\varkappa + 2\delta)] \tag{4.10}$$

the sum is asymptotically equal to the term

$$Z(\varkappa) \exp[i\varkappa x + tp(\varkappa) + 2nt\gamma(\varkappa)]$$

in which $k_1 = \ldots = k_{2n+1} = \varkappa$. When t is sufficiently large, a periodic motion whose wavelength is equal to $\varkappa$ is set up in the system

$$X_{11} = Ze^{i\varkappa x}\left[A\varepsilon e^{pt} \sum_{n=0}^{\infty} f_n \left(\frac{\varepsilon^2 |A|^2 e^{2\gamma t}}{2\gamma}\right)^n\right] \tag{4.11}$$

Each term in the sum in (4.11) grows exponentially, although the whole sum may remain finite at any t. The function of time appearing within the square brackets in (4.11) represents a solution of the initial problem

$$dQ / dt = pQ + \Gamma Q^2 Q^*, Q(0) = \varepsilon A, \quad p = \gamma + i\Omega, \ \Gamma = B + iD, \ \gamma > 0 \tag{4.12}$$

for large t. Indeed, after the substitution

$$Q = Re^{pt}, \qquad T = (e^{2\gamma t} - 1) / (2\gamma)$$

the solution of (4.12) is easily obtained either directly, or employing the series (4.2). In the first case the solution is

$$Q = \varepsilon A e^{pt} (1 - 2TB\varepsilon^2 |A|^2)^{-1/2\Gamma / B} \tag{4.13}$$

while in the second case we have

$$Q = \varepsilon A e^{pt} \sum_{n=0}^{\infty} f_n (\varepsilon^2 |A|^2 T)^n \tag{4.14}$$

which coincides with the function of time in (4.11) when $\gamma t \gg 1$. Comparing (4.13) and (4.14) we see that the series in (4.14) converges(*) if

$$2TB\varepsilon^2 |A|^2 < 1$$

$$f_n = \Gamma (2\Gamma + \bar{\Gamma}) \ldots [n\Gamma + (n-1)\bar{\Gamma}] / n!$$

If $B < 0$ we have, for $t \to \infty$,

$$Q \to \sqrt{q_\infty} \exp it (\Omega + Dq_\infty), \qquad q_\infty = -\gamma / B$$

5. The results obtained above are easily generalized to the case of the exact Eq. (2.15). Again we obtain the expression (4.6) in which

$$nf_n = \sum_{i=1}^{n} \Gamma_i \sum_{m} f_{m_1} \ldots f_{m_{i+1}} \bar{f}_{m_{i+2}} \ldots \bar{f}_{m_{2i+1}} \qquad (m_1 + \ldots + m_{2i+1} = n - 1) \tag{5.1}$$

and where all wave numbers entering Γ are equal to $\varkappa$. Inserting (4.6) and (4.2) into the sum

$$X_{n, n+2m} = \sum_{k \approx nk_0} e^{ikx} \Sigma Y_{n, n+2m} Q_1 \ldots Q_{n+m} \bar{Q}_{n+m+1} \ldots \bar{Q}_{n+2m} \tag{5.2}$$

we find, that any product of $n + 2m$ magnitudes $Q^{(i)}$ tends asymptotically to the term, in which all wave numbers are equal to $\varkappa$. Retaining only these terms we obtain

$$X_{n, n+2m} = Y_{n, n+2m} (k_1 = \ldots = k_{n+2m} = \varkappa) \, e^{in\varkappa x} Q^{n+m} \bar{Q}^m$$

where Q is a function of time given by (4.11), with the coefficients f given by (5.1). We find that this function represents the solution of (3.1) for $k = \varkappa$, $Q(0) = \varepsilon A(\varkappa)$ and $\gamma t \gg 1$, and we can easily confirm it by obtaining a solution of (3.1) in the form of (4.2).

Thus, when t is sufficiently large (boundedness of the system becomes then essential), a periodic motion of wavelength $\varkappa$ is set up in the system.

This motion is described by (2.14) and (2.15) in which only $Q(\varkappa) \neq 0$; such relations can be obtained using the methods given in [2, 3 and 5], assuming that the wave number (which becomes the undefined parameter in all these methods) is equal to that value of $\varkappa$, for which the increment of the discrete spectrum has the largest value.

Behavior of the system at large t can also be studied by obtaining the solution X of Eqs. (2.1) in the form of a series in terms of the initial amplitude ε and summing the terms containing the same factor $\exp(ikx)$.

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*) When $t \to \infty$, we continue the solution which has the form of a power series, analytically into the region of divergence of the series. Similar procedure is widely applied in statistics and quantum field theory.

Let us now investigate how the steady-state motion of the system depends on the parameter when (Fig. 1) k_0 is a function of λ

$$k_0(\lambda) \approx k_* + k_*'(\lambda - \lambda_*), \qquad k_*' \neq 0 \tag{5.3}$$

In accordance with (1.2), in this case there exists a discrete set of values λ_n for which the increments of the discrete spectrum achieve the maximum at two points $k_0 \pm \delta$. From (5.3) it follows that

$$\lambda_{n+1} - \lambda_n = G_1 / l, \qquad G_1 = 2\delta l / |k_*'| \tag{5.4}$$

Thus the wave number $\varkappa$ of the periodic motion will remain constant within the intervals $\lambda_n < \lambda < \lambda_{n+1}$ and will change discontinuously by 2δ when λ passes through λ_n. If $k_*' > 0$, then $\varkappa$ increases together with increasing supercriticality $\lambda - \lambda_*$.

Discontinuous change of the wave number is acompanied by the discontinuous change in the amplitude of the steady-state motion. Let us denote

$$q = q(\varkappa, \lambda_n), \qquad q_+ = q(\varkappa + 2\delta, \lambda_n)$$

Relation (3.2) yields, with accuracy of the order of $\sim\delta$,

$$q_+ - q = 2\delta \; [(\partial S / \partial k) / (\partial S / \partial q)]_{\varkappa, \lambda_n}$$

where S denotes the sum in (3.2). If $\gamma_1(k_*, \lambda_*) < 0$, then q is small when the value of the supercriticality is small and

$$(q_+ - q) / q = G_2/l, \; G_2 = 2\delta l \; (\gamma_1^{-1}\partial\gamma_1/\partial k)_{k_*, \gamma_*} \tag{5.5}$$

the amplitude is approximately given by (3.2) in which $k = k_0(\lambda)$:

$$q(\varkappa, \lambda) \approx q(k_0, \lambda) \; [1 + (\varkappa - k_0) G_2 / (2\delta l)], \qquad |\varkappa - k_0| \leqslant \delta$$

Discontinuous amplitude changes given by (5.5) take place whenever λ passes through λ_n irrespective of the direction of change of λ, unlike the changes which take place under the impulsive excitation [4 and 5].

When $\lambda = \lambda_n$, the steady-state motion of the system ceases to be periodic and becomes turbulent; it is then described by expressions (2.14) and (2.15) in which only $Q_\pm = Q(k_0 \pm \pm\delta)$ differ from zero. Indeed, the second sum of (4.9) is asymptotically equal to the sum of those terms, in which $k_i = k_0 \pm \delta$, $i = 1, \ldots, 2n+1$, and where the values of the factors preceding the second sum can be taken as those at $k = k_0$. Then the expression (4.9) becomes (4.1) where Q is the solution of the initial problem

$$\frac{dQ}{dt} = p(k)\,Q + \Gamma \Sigma Q_1 Q_2 \bar{Q}_3, \qquad Q_\pm(0) = \varepsilon A(k_0) \tag{5.6}$$

$$Q(0, k \neq k_0 \pm \delta) = 0, \qquad \Gamma = \Gamma_1(k_0, k_0, k_0) = B + iD, \qquad B < 0$$

at the large values of t. Steady-state solution of (5.6) is

$$Q(k \neq k_0 \pm \delta) = 0, \qquad Q_\pm = \sqrt{q}\exp it(\Omega_\pm + 3qD), \qquad q = -{}^1/_3\gamma / B$$

and in the steady-state (5.2) represents a wave packet with the wave numbers given by

$$k = nk_0 + (N - 2i)\,\delta, \qquad N = n + 2m, \qquad i = 0, 1, \ldots, N$$

The bandwidth of this packet is equal to $2\delta N$. Beginning from $N \approx k_0/\delta$, the wave packets appearing in the sum

$$X_N = \sum_{i=0}^{N} X_{N-2i,N} \sim |Q|^N, \qquad X_{-n,N} = \bar{X}_{n,N}$$

merge together, filling the whole range of the wave numbers $(-Nk_0, Nk_0)$. Since

$$X = X_1 + X_2 + \ldots + X_N + \ldots$$

it follows that in the steady-state pulsations may occur, which can be of any scale. At large λ_n turbulence develops.

If $\lambda \approx \lambda_n$, then the turbulent motion described here exists for some t when $|\gamma_0''|\delta^2 t \gg 1$ and the inequality (4.10) has the opposite sense (using the terminology of [12] we can say that two degrees of freedom are excited in this turbulent motion).

6. In one-dimensional problems discussed above, k and x were pure numbers. When considering problems of motion of a medium between horizontal planes, we must take k and x as vectors and assume the product kx to be a scalar product

$$kx = k^1 x_1 + k^2 x_2, \qquad |x_{1,2}| < {}^1/_2 \; l$$

In this case the vertical coordinate x_3 will be the transverse one, and the distance between the planes will be assumed very small compared with the horizontal dimension l.

Increments γ of perturbations of plane-parallel flows in the x_1-direction will be the largest [11] when $k^2 = 0$, $k^1 \neq 0$. Therefore we can reduce (2.5) and (2.6) to (2.14) and (2.15), provided that the supercriticality is small.

Let $k_* = (k_*^1, 0)$ be a vector for which $\gamma(k_*, \lambda_*) = 0$. If in (2.15)

$$B = \mathrm{Re}\ \Gamma_1 (k_1 = k_2 = k_3 = k_*,\ \lambda_*) < 0$$

then, provided that the supercriticality is small, a one-dimensional motion ($k^2 = 0$) of a small amplitude will be set up in the system.

When $B > 0$, a large amplitude motion takes place in the system, which may be neither one-dimensional, nor periodic. It should disappear in the region $\lambda < \lambda_*$ with decreasing supercriticality, when the infinitesimal perturbations decay [4 and 5].

In the problem dealing with the onset of convection between two horizontal planes the magnitude $\gamma = \gamma(|k|)$, therefore a large number of perturbations whose wave vectors are equal in their moduli accumulates in the system, even when the supercriticality is arbitrarily small. Unlike the problem on plane-parallel flows, the latter problem is basically two-dimensional and needs a separate consideration.

7. In conclusion we may note that, when the system is bounded, the steady-state motion has the wavelength for which the linear increment is largest. In some cases this also applies to infinite systems. If, however, the instability of an infinite system is removable, then (see Appendix) the system returns to the state of equilibrium.

Discreteness of the wave numbers imitates the discreteness of the spectrum of the system, when the boundary conditions at $x = \pm \frac{1}{2} l$ are taken into account. Eigenfunctions $f_n(x, r)$ of this spectrum corresponding to the accumulating perturbations, can be characterized by the number of extrema in x; this number increases with n. When n is large, the dependence of f_n on x can be isolated (except in the end regions) in the form of $\exp(inc_n x/l)$ where $c_n \sim 1$ may depend on n; thus the right-hand sides of (5.4) and (5.5) may also depend on n.

A p p e n d i x. We have said before that, when $t\delta^2|\gamma_0''| \ll 1$, then the bounded system behaves like an infinite one; in this case summation over k can be replaced everywhere by integration as e.g.

$$\sum fQ_1 \ldots Q_n \to \int fQ_1' \ldots Q_n' \delta(k_1 + \ldots + k_n - k)\, dk_1 \ldots dk, \qquad Q' = {}^1/_2\, Q/\delta$$

The second sum in (4.9) can be replaced by the product of integrals of the type

$$J(x, t) = \int Z(k) \exp[ikx + tp(k)]\, dk \tag{A.1}$$

Behavior of this integral at large t determines the type of instability of the equilibrium state. We shall call the instability absolute when $|J| \to \infty$ as $t \to \infty$ and removable, when $J \to 0$ [12]. Generally speaking, we must know how $p(k)$ behaves in the complex k-plane, before we can obtain an estimate for J. The case given below when the group velocity $v = \Omega_0'$ in the expansion

$$p(k) = p_0 + (k - k_0)\ iv + {}^1/_2\ p_0''\ (k - k_0)^2 + \ldots$$

is small, is an exception. Group velocity can be small in those systems, in which $v = 0$ for some λ. When v is small, a saddle point (defined by $p' = 0$) exists near $k = k_0$ and we have

$$J = (-2\pi / tp'')^{1/2} Z(k) e^{ikx+pt} \qquad (t \to \infty) \tag{A.2}$$

$$k \approx k_0 - iv / p_0'' = \alpha + i\beta, \quad p \approx p_0 + {}^1/_2\, v^2 / p_0'' = \gamma + i\Omega, \quad p'' \approx p_0''$$

Let us assume that the curves $\gamma_0 = v = 0$ intersect in the $\lambda\mu$-plane (Fig. 2). By (A.2) the instability is absolute if

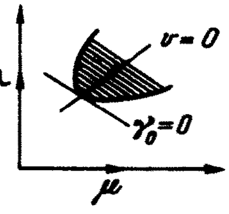

Fig. 2

$$\gamma = \gamma_0 + {}^1/_2\gamma_0''\, v^2\, |p_0''|^2 > 0$$

Let the parameters λ and μ be such, that the instability is absolute (the corresponding region in Fig. 2 is shaded). Then from (4.6), (4.9) and (A.2) we have

$$X_{11} = \sum_{n>0} A\varepsilon \,|\, A\varepsilon\,|^{2n} f_n Z(k + 2ni\beta) \left(-\frac{2\pi}{tp''}\right)^{1/2} \left(\frac{2\pi}{t\,|\,p''\,|}\right)^n \exp\,[ix\,(k + 2ni\beta) + (p + 2n\gamma)\,t] \tag{A.3}$$

where $A = A(k)$ while f_n is given by (4.8) in which Γ/n is replaced by

$$\Gamma\,(k,\ k,\ k)\ /\ [2n\gamma + p\,(k) - p\,(k + 2ni\beta)] \tag{A.4}$$

and the real part of the denominator is assumed positive

$$n < n_0 \sim -\gamma_0 / (\gamma_0''\beta^2)$$

If we make the substitution $k + 2ni\beta \approx k$ in the expression for f_n and Z, then (A.3), (A.4), (4.13) and (4.14) yield readily

$$X_{11} \approx Z(k)\, e^{ikx+pt} A\varepsilon \left(\frac{2\pi}{-tp''}\right)^{1/2} \left(1 - B\,\frac{|\,A\varepsilon\,|^2\, 2\pi \exp\,(2\gamma t - 2\beta x)}{\gamma t\,|\,p''\,|}\right)^{-{}^1/_2\Gamma/B} \tag{A.5}$$

which becomes exact when $v = 0$. When $B < 0$, we have

$$X_{11} \to Z\,\sqrt{q}\,\exp\ i\ [\alpha x + t\,(\Omega + qD)], \qquad q = -\gamma / B$$

In the case of removable instability, nonlinear effects are unimportant at any t provided that the initial deviation from equilibrium is sufficiently small. Return of the system to the equilibrium state can be adequately described by linearized equations.

Fig. 2 shows that, when the supercriticality is sufficiently small, then the instability is removable provided that $v \neq 0$ at the boundary of stability. This remains true even for larger values of v, since the decrement γ increases together with v. In a coordinate system moving with velocity $u \approx v$, the instability is removable when $\gamma(v - u) \leqslant 0$ (an estimate for $J(x - ut,\ t)$ can be obtained from (A.2) by making the substitution $v \to (v - u)$ and multiplying the result by $\exp(-ik_0 ut)$). When $u = v$, then $X_{11}(x - vt,\ t)$ is given (with the accuracy of up to the factor $\exp(-ik_0 vt)$) by (A.5), in which $v = 0$. We must note that this expression cannot be used in the fixed coordinate system; similarly to the expression (A.2) when $v = 0$, it is only valid for

$$|\,x\,| < (\Delta x) \sim 1/(\Delta k) \sim \sqrt{t\,|\,p_0''\,|} \tag{A.6}$$

where (Δk) is the effective bandwidth of the spectrum. In the region $|x| > (\Delta x)$, oscillations of J decay exponentially with increasing $|x|$. The fixed point of the immovable coordinate system moves, in this case, with the velocity equal to v, therefore (A.6) does not hold when $t \to \infty$. This makes possible the assertion, that, when $v \neq 0$ and the instability is absolute, then the perturbation initially increases according to (A.3 and A.5) and then disappears.

In a bounded system the time of motion of the packet $t \sim 1/(\delta v)$ is finite, therefore (A.6) holds when $x \sim l$ provided that $|p_0''|\ \delta v > 1$. Discreteness of the spectrum becomes, however, important and (4.11) with (4.13) should be used instead of (A.5).

Thus, in the case of removable instability, the appearance of periodic motion is governed by the boundedness of the system.

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